

Translated points and Rabinowitz Floer homology

Peter Albers and Will J. Merry

Abstract

We prove that if a contact manifold admits an exact filling then every local contactomorphism isotopic to the identity admits a translated point in the interior of its support, in the sense of Sandon [San11b]. In addition we prove that if the Rabinowitz Floer homology of the filling is non-zero then every contactomorphism isotopic to the identity admits a translated point, and if the Rabinowitz Floer homology of the filling is infinite dimensional then every contactomorphism isotopic to the identity has either infinitely many translated points, or a translated point on a closed leaf. Moreover if the contact manifold has dimension greater than or equal to 3, the latter option generically doesn't happen. Finally, we prove that a generic contactomorphism on \mathbb{R}^{2n+1} has infinitely many geometrically distinct iterated translated points all of which lie in the interior of its support.

1 Introduction

Let (Σ^{2n-1}, ξ) denote a coorientable closed contact manifold, and let α denote a 1-form on Σ such that $\xi = \ker \alpha$. Let R_α denote the Reeb vector field of α , and let $\phi_t^\alpha : \Sigma \rightarrow \Sigma$ denote the flow of R_α .

Denote by $\text{Cont}(\Sigma, \xi)$ the group of contactomorphisms $\psi : \Sigma \rightarrow \Sigma$, and denote by $\text{Cont}_0(\Sigma, \xi) \subseteq \text{Cont}(\Sigma, \xi)$ those contactomorphisms ψ that are contact isotopic to $\mathbb{1}$.

Definition 1.1. Fix $\psi \in \text{Cont}(\Sigma, \xi)$, and write $\psi^*\alpha = \rho\alpha$ for $\rho \in C^\infty(\Sigma, \mathbb{R}^+)$. We say that a point $x \in \Sigma$ is called a **translated point** for ψ if there exists $\tau \in \mathbb{R}$ such that

$$\psi(x) = \phi_\tau^\alpha(x) \quad \text{and} \quad \rho(x) = 1.$$

We say that a point $x \in \Sigma$ is called a **iterated translated point** for ψ if it is a translated point for some iteration ψ^n .

The notion of (iterated) translated points was introduced by Sandon in [San11b] and further explored in [San11a]. We refer to the reader to these papers for a discussion as to why translated points are a worthwhile concept to study.

Let ξ_{st} denote the standard contact structure on \mathbb{R}^{2n-1} . Suppose $\sigma : \mathbb{R}^{2n-1} \rightarrow \mathbb{R}^{2n-1}$ is a contactomorphism such that $\mathfrak{S}(\sigma) := \text{supp}(\sigma - \mathbb{1})$ is compact, and suppose that $\mathbf{x} : \mathbb{R}^{2n-1} \rightarrow U \subseteq \Sigma$ is a Darboux chart onto an open subset U of Σ . Then we can form a contactomorphism $\psi : \Sigma \rightarrow \Sigma$ such that $\psi = \sigma \circ \mathbf{x}$ on U and $\psi = \mathbb{1}$ on $\Sigma \setminus U$. We call ψ the **local contactomorphism** induced from σ . In this case we are only interested in translated points of ψ in the interior of $\mathfrak{S}(\psi)$. Indeed, if $x \in \Sigma \setminus \mathfrak{S}(\psi)$ is any periodic point for the Reeb flow ϕ_t^α then x is vacuously a translated point of ψ .

Remark 1.2. Since a ball of arbitrary radius in \mathbb{R}^{2n-1} is contactomorphic to \mathbb{R}^{2n-1} any contactomorphism of \mathbb{R}^{2n-1} gives rise to a local contactomorphism via an appropriate Darboux chart.

Definition 1.3. We say that $(\Sigma, \xi = \ker \alpha)$ admits an **exact filling** if there exists a compact symplectic manifold $(M, d\lambda_M)$ such that $\Sigma := \partial M$ and such that $\alpha = \lambda_M|_\Sigma$.

In this case let us denote by $X := M \cup_\Sigma \{\Sigma \times [1, \infty)\}$ the **completion** of M . Define

$$\lambda := \begin{cases} \lambda_M, & \text{on } M, \\ r\alpha, & \text{on } \Sigma \times [1, \infty). \end{cases}$$

Then $(X, d\lambda)$ is an exact symplectic manifold that is convex at infinity, and $\Sigma \subseteq X$ is a hypersurface of restricted contact type.

Since X is an exact symplectic manifold that is convex at infinity and Σ is a hypersurface of restricted contact type, the **Rabinowitz Floer homology** of the pair (Σ, X) is a well defined \mathbb{Z}_2 -vector space. Rabinowitz Floer homology was discovered by Cieliebak and Frauenfelder in [CF09], and has since generated many applications in symplectic topology (we refer to the survey article [AF10b] for more information on Rabinowitz Floer homology).

We can now state our main result.

Theorem 1.4. *Suppose (Σ, ξ) is a closed contact manifold admitting an exact filling $(M, d\lambda_M)$. Then:*

1. *If $\psi \in \text{Cont}_0(\Sigma, \xi) \setminus \{\mathbb{1}\}$ is a local contactomorphism then ψ has translated point $x \in \text{int}(\mathfrak{S}(\psi))$.*
2. *If the Rabinowitz Floer homology $\text{RFH}(\Sigma, X)$ does not vanish then every $\psi \in \text{Cont}_0(\Sigma, \xi)$ has a translated point.*
3. *If $\text{RFH}(\Sigma, X)$ is infinite dimensional then for $\psi \in \text{Cont}_0(\Sigma, \xi)$ either ψ has infinitely many translated points or ψ has a translated point lying on a closed leaf of R_α .*
4. *If $\dim \Sigma \geq 3$ then a generic $\psi \in \text{Cont}_0(\Sigma, \xi)$ has no translated point lying on a closed leaf of R_α .*
5. *For a generic $\psi \in \text{Cont}_0(\Sigma, \xi)$ the following holds. If $x \in \Sigma$ is a translated point for ψ^n , $n \in \mathbb{N}$, then x is **not** a translated point for $\psi, \psi^2, \dots, \psi^{n-1}$.*

Remark 1.5. Property 5. in Theorem 1.4 holds in fact for leafwise intersections as the proof will show.

The following corollary is well-known and follows from Chekanov's work [Che96].

Corollary 1.6. *Any $\psi \in \text{Cont}_0(\mathbb{R}^{2n-1}, \xi_{\text{st}}) \setminus \{\mathbb{1}\}$ admits a translated point $x \in \text{int}(\mathfrak{S}(\psi))$.*

Proof. This follows from Theorem 1.4 together with Remark 1.2. ■

Corollary 1.7. *A generic $\psi \in \text{Cont}_0(\mathbb{R}^{2n-1}, \xi_{\text{st}}) \setminus \{\mathbb{1}\}$ admits infinitely many geometrically distinct iterated translated points all of which lie in $\text{int}(\mathfrak{S}(\psi))$.*

Proof. By the previous corollary every ψ^n admits a translated point $x_n \in \text{int}(\mathfrak{S}(\psi^n))$. By property 5. in Theorem 1.4 the set $\{x_n : n \in \mathbb{N}\}$ cannot be finite for a generic ψ . ■

Remark 1.8. Sandon proved in [San11a] that any **positive** ψ admits infinitely many geometrically distinct translated points.

In order to explain the idea behind the proof of Theorem 1.4, we need to introduce a few more definitions. Recall that from (Σ, ξ) we can build the **symplectization** of Σ , which is the exact symplectic manifold $(S\Sigma, d(r\alpha))$, where

$$S\Sigma := \Sigma \times \mathbb{R}^+,$$

and r is the coordinate on $\mathbb{R}^+ := (0, \infty)$. Suppose $\psi \in \text{Cont}(\Sigma, \xi)$. There exists a unique positive smooth function $\rho \in C^\infty(\Sigma, \mathbb{R}^+)$ such that $\psi^*\alpha = \rho\alpha$. We define the **symplectization** of ψ to be the symplectomorphism $\varphi : S\Sigma \rightarrow S\Sigma$ defined by

$$\varphi(x, r) = (\psi(x), r\rho(x)^{-1}).$$

Let us now go back to the completion X of M . Let Y_M denote the Liouville vector field of λ_M (defined by $i_{Y_M}d\lambda_M = \lambda_M$). The entire symplectization $S\Sigma$ embeds into X via the flow of Y_M , and under this embedding the vector field Y on X defined by

$$Y := \begin{cases} Y_M, & \text{on } M, \\ r\partial_r, & \text{on } S\Sigma \end{cases}$$

satisfies $i_Y d\lambda = \lambda$ on all of X . Note that under this embedding $S\Sigma \hookrightarrow X$, the hypersurface $\Sigma \times \{1\}$ in $S\Sigma$ is identified with Σ in X .

Suppose $\varphi \in \text{Symp}(X, \omega)$. A point $x \in \Sigma$ is called a **leaf-wise intersection point** for (Σ, φ) if there exists $\tau \in \mathbb{R}$ such that

$$\varphi(x) = \phi_\tau^\alpha(x).$$

The definition still makes sense if φ is only defined on $S\Sigma \subseteq X$ rather than on all of X . In this case it is more convenient to phrase the definition using the hypersurface $\Sigma \times \{1\}$ inside $S\Sigma$. Thus if $\varphi \in \text{Symp}(S\Sigma, d(r\alpha))$ then a point $x \in \Sigma$ is called a leaf-wise intersection point for (Σ, φ) if there exists $\tau \in \mathbb{R}$ such that

$$\varphi(x, 1) = (\phi_\tau^\alpha(x), 1).$$

Our starting point is the following observation of Sandon [San11b].

Lemma 1.9. *Fix $\psi \in \text{Cont}(\Sigma, \xi)$ and let $\varphi \in \text{Symp}(S\Sigma, d(r\alpha))$ denote the symplectization of ψ . Then a point $x \in \Sigma$ is a translated point for ψ if and only if $(x, 1)$ is a leaf-wise intersection point for φ .*

This reduces the existence problem for translated points of ψ to the existence problem of leaf-wise intersections for φ . The first and second authors, developed in [AF10a] a variational characterization for the leaf-wise intersection problem using the Rabinowitz action functional, and it is precisely this characterization that we will exploit.

Acknowledgement. We thank Urs Frauenfelder and Sheila Sandon for several useful comments.

2 Proofs

Fix once and for all a 1-form $\alpha \in \Omega^1(\Sigma)$ such that $\xi = \ker \alpha$, and fix an exact symplectic filling $(M, d\lambda_M)$. As usual denote the completion of $(M, d\lambda_M)$ by $(X, d\lambda)$. We denote by

$$\wp(\Sigma, \alpha) > 0 \tag{2.1}$$

the minimal period of an orbit of R_α that is contractible in X .

We denote by $\mathcal{C}(\Sigma, \xi)$ the set of contact isotopies $\{\psi_t\}$ for $t \in \mathbb{R}$ which satisfy $\psi_0 = \mathbb{1}$ and $\psi_{t+1} = \psi_t \circ \psi_1$. The universal cover $\widehat{\text{Cont}}_0(\Sigma, \xi)$ of $\text{Cont}_0(\Sigma, \xi)$ consists of equivalence classes of members of $\mathcal{C}(\Sigma, \xi)$, where two paths $\{\psi_t\}$ and $\{\psi'_t\}$ are equivalent if there exists a smooth family $\{\psi_{s,t}\}$ for $(s, t) \in [0, 1] \times \mathbb{R}$ such that $\psi_{0,t} = \psi_t$ and $\psi_{1,t} = \psi'_t$ with $\{\psi_{s,t}\} \in \mathcal{C}(\Sigma, \xi)$ for each $s \in [0, 1]$.

The infinitesimal generator W of a contact isotopy $\{\psi_t\} \in \mathcal{C}(\Sigma, \xi)$ is defined by

$$W(x) := \left. \frac{\partial}{\partial t} \right|_{t=0} \psi_t(x),$$

and we say that $\{\psi_t\}$ is **generated** by the function $h : \mathbb{R}/\mathbb{Z} \times \Sigma \rightarrow \mathbb{R}$ defined by

$$h(t, x) := \alpha_{\psi_t(x)}(W(\psi_t(x))) \tag{2.2}$$

(h is 1-periodic because $\psi_{t+1} = \psi_t \circ \psi_1$).

Suppose $\{\psi_t\} \in \mathcal{C}(\Sigma, \xi)$. In this case if $\psi := \psi_1$ then the symplectization φ of ψ belongs to $\text{Ham}(S\Sigma, d(r\alpha))$. Indeed, if we define the **contact Hamiltonian** $H : \mathbb{R}/\mathbb{Z} \times S\Sigma \rightarrow \mathbb{R}$ associated to $\{\psi_t\}$ by

$$H(t, x, r) := rh(t, x),$$

where h is the function from (2.2), and we denote by φ_t the Hamiltonian flow of H , then it can be shown (see for instance [AF11, Proposition 2.3]) that

$$\varphi_t(x, r) = (\psi_t(x), r\rho_t(x)^{-1}).$$

Thus the symplectization φ of ψ is simply φ_1 . We define

$$F_0 : S\Sigma \rightarrow \mathbb{R}$$

by

$$F_0(x, r) := f(r),$$

where

$$\begin{aligned} f(r) &:= \frac{1}{2}(r^2 - 1) \quad \text{on } (1/2, \infty), \\ f''(r) &\geq 0 \quad \text{for all } r \in \mathbb{R}^+, \\ \lim_{r \rightarrow 0} f(r) &= -\frac{1}{2} + \varepsilon \end{aligned} \tag{2.3}$$

for some small $\varepsilon > 0$. Note that the Hamiltonian vector field X_{F_0} is given by $X_{F_0}(x, r) = f'(r)R_\alpha(x)$; in particular $X_{F_0}|_{\Sigma \times \{1\}} = R_\alpha$.

Let $\chi \in C^\infty(S^1, [0, \infty))$ denote a smooth function such that if

$$\bar{\chi}(t) := \int_0^t \chi(s) ds$$

then there exists $t_0 \in (0, 1/2)$ such that $\bar{\chi}(t) \equiv 1$ on $[t_0, 1]$, and such that on $[0, t_0]$ the function $\bar{\chi}$ is strictly increasing.

Finally fix a smooth function $\vartheta : [0, 1] \rightarrow [0, 1]$ such that $\vartheta(t) = 0$ for $t \in [0, 1/2]$, and such that $\vartheta(1) = 1$ with $0 \leq \dot{\vartheta}(t) \leq 4$ for all $t \in [0, 1]$. Denote by $\mathcal{L}(S\Sigma) := C^\infty(S^1, S\Sigma)$. We now define the Rabinowitz action functional we will work with.

Definition 2.1. We define the **Rabinowitz action functional**

$$\mathcal{A} : \mathcal{L}(S\Sigma) \times \mathbb{R} \rightarrow \mathbb{R}$$

by

$$\mathcal{A}(v, \eta) := \int_0^1 v^* \lambda - \eta \int_0^1 \chi(t) F_0(v) dt - \int_0^1 \dot{\vartheta}(t) H(\vartheta(t), v) dt.$$

A simple calculation tells us that if $(v, \eta) \in \text{Crit}(\mathcal{A})$ then if we write $v(t) = (x(t), r(t))$ we have

$$\begin{cases} \dot{v}(t) = \eta \chi(t) X_{F_0}(v) + \dot{\vartheta}(t) X_H(\vartheta(t), v), \\ \int_0^1 \chi(t) F_0(v) dt = 0. \end{cases}$$

The following lemma appears in [AF10a, Proposition 2.4], and explains the connection between the Rabinowitz action functional and leaf-wise intersection points (and hence translated points, via Lemma 1.9).

Lemma 2.2. *Define*

$$e : \text{Crit}(\mathcal{A}) \rightarrow \Sigma$$

by

$$e(v, \eta) := x(1/2)$$

where $v = (x, r)$. Then e is a surjection onto the set of translated points for ψ . If ψ has no translated points lying on closed leaves of R_α then e is a bijection. Moreover

$$\mathcal{A}(v, \eta) = \eta$$

for $(v, \eta) \in \text{Crit}(\mathcal{A})$.

Proof. Suppose $(v, \eta) \in \text{Crit}(\mathcal{A})$. Write $v(t) = (x(t), r(t)) \in \Sigma \times \mathbb{R}^+$. For $t \in [0, 1/2]$ one has $\dot{v}(t) = \eta \chi(t) X_{F_0}(v)$. Since $F_0 = \chi f$ with f autonomous, F_0 is constant on its flow lines and thus $t \mapsto F_0(v(t)) = r(t)$ is constant for $t \in [0, 1/2]$. The second condition tells this constant is 1, and hence $v(1/2) = (\phi_\tau^\alpha(x(0)), 1)$.

Next, for $t \in [1/2, 1]$ we have $\dot{v}(t) = \dot{\vartheta}(t) X_H(\vartheta(t), v)$. Thus $v(t) = \varphi_{\vartheta(t)}(v(1/2))$ for $t \in [1/2, 1]$. In particular, $\varphi(v(1/2), 1) = (\phi_\tau^\alpha(v(1/2)), 1)$, and thus $v(1/2)$ is a leaf-wise intersection point of φ .

In order to prove the last statement, we first note that

$$\lambda(X_{F_0}(x, r)) = f'(r) \alpha_x(R_\alpha(x)) = f'(r),$$

$$\lambda(X_H(x, r)) = dH(x, r)(r \partial_r) = H(x, r),$$

and hence

$$\begin{aligned} \mathcal{A}(v, \eta) &= \int_0^{1/2} \lambda(\eta \chi X_{F_0}(v)) dt + \int_{1/2}^1 \left[\lambda(\dot{\vartheta} X_H(\vartheta(t), v)) - \dot{\vartheta} H(\vartheta(t), v) \right] dt \\ &= \eta + 0. \end{aligned}$$

■

Unfortunately, in order to be able to define the Rabinowitz Floer homology, we cannot work with \mathcal{A} as it is not defined on all of $\mathcal{L}X \times \mathbb{R}$. In order to rectify this, we extend F_0 and H to Hamiltonians defined on all of X . Here are the details. Define

$$F : X \rightarrow \mathbb{R}$$

by setting

$$F|_{X \setminus S\Sigma} := -1/2 + \varepsilon,$$

where $\varepsilon > 0$ is as in (2.3), and defining $F = F_0$ on $S\Sigma$. Next, for $c > 0$ let $\beta_c \in C^\infty([0, \infty), [0, 1])$ denote a smooth function such that

$$\beta_c(r) = \begin{cases} 1, & r \in [e^{-c}, e^c], \\ 0, & r \in [0, e^{-2c}] \cup [e^c + 1, \infty), \end{cases}$$

and such that

$$\begin{aligned} 0 &\leq \dot{\beta}_c(r) \leq 2e^{2c} \quad \text{for } r \in [e^{-2c}, e^{-c}], \\ -2 &\leq \dot{\beta}_c(r) \leq 0 \quad \text{for } r \in [e^c, e^c + 1]. \end{aligned}$$

Then define $H_c : [0, 1] \times X \rightarrow \mathbb{R}$ by

$$H_c|_{[0, 1] \times (X \setminus S\Sigma)} := 0,$$

and for $(t, x, r) \in [0, 1] \times S\Sigma$,

$$H_c(t, x, r) := \beta_c(r) r \dot{\vartheta}(t) h(\vartheta(t), x).$$

Remark 2.3. Note that for any $c > 0$, H_c is a compactly supported 1-periodic Hamiltonian on X with the property that $H_c(t, \cdot, \cdot) = 0$ for $t \in [0, 1/2]$. Moreover the **Hofer norm** $\|H_c\|$ of H_c satisfies

$$\|H_c\| \leq 4(e^c + 1)(h_+ + h_-),$$

where

$$h_+ := \max_{(t, x) \in \mathbb{R}/\mathbb{Z} \times \Sigma} h(t, x), \quad h_- := - \min_{(t, x) \in \mathbb{R}/\mathbb{Z} \times \Sigma} h(t, x).$$

Remark 2.4. Suppose $\sigma : \mathbb{R}^{2n-1} \rightarrow \mathbb{R}^{2n-1}$ is a contactomorphism such that $\mathfrak{S}(\sigma)$ is compact, and suppose that $\mathbf{x} : \mathbb{R}^{2n-1} \rightarrow U \subseteq \Sigma$ is a Darboux chart onto an open subset U of Σ . Let $\psi : \Sigma \rightarrow \Sigma$ denote the local contactomorphism such that $\psi = \sigma \circ \mathbf{x}$ on U and $\psi = \mathbb{1}$ on $\Sigma \setminus U$. Given $R > 0$, let $\tau_R : \mathbb{R}^{2n-1} \rightarrow \mathbb{R}^{2n-1}$ denote the contact rescaling defined by $\tau_R(\mathbf{x}, \mathbf{y}, z) = (R\mathbf{x}, R\mathbf{y}, R^2z)$ for $(\mathbf{x}, \mathbf{y}, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}$. There is a 1-1 correspondence between the translated points of σ and the translated points of the conjugation $\sigma_R := \tau_R \circ \sigma \circ \tau_R^{-1}$ as follows: $(\mathbf{x}, \mathbf{y}, z)$ is a translated point of σ if and only if $\tau_R(\mathbf{x}, \mathbf{y}, z)$ is translated point of σ_R . Moreover if σ is generated by the function $h(t, x)$ then σ_R is generated by $R^2h(t, \tau_R^{-1}(x))$.

We denote by ψ_R the local contactomorphism of Σ corresponding to σ_R and the function $H_{c,R}$ corresponding to ψ_R . Then for fixed $c > 0$ we can choose R so small that the Hofer norm of $H_{c,R}$ is smaller than $\varphi(\Sigma, \alpha)$.

We now extend the Rabinowitz action functional \mathcal{A} to a new functional

$$\mathcal{A}_c : \mathcal{L}X \times \mathbb{R} \rightarrow \mathbb{R}$$

by

$$\mathcal{A}_c(v, \eta) := \int_0^1 v^* \lambda - \eta \int_0^1 \chi(t) F_0(v) dt - \int_0^1 H_c(t, v) dt.$$

The following result is the key to the present paper. The proof is similar to (but simpler than) [AF11, Proposition 4.3].

Proposition 2.5. *There exists $c > 0$ such that if $c > c_0$ then if $(v, \eta) \in \text{Crit}(\mathcal{A}_c)$ then $v(S^1) \subseteq \Sigma \times \mathbb{R}^+$, and moreover if we write $v(t) = (x(t), r(t))$ then $r(S^1) \subseteq (e^{-c/2}, e^{c/2})$.*

Proof. We know that $r(t) = 1 \in (e^{-c/2}, e^{c/2})$ for all $t \in [0, 1/2]$. Thus if

$$I := \left\{ t \in S^1 : r(t) \in (e^{-c/2}, e^{c/2}) \right\}$$

then I is a non-empty open interval containing the interval $[0, 1/2]$. Let $I_0 \subseteq I$ denote the connected component containing 0. We show that I_0 is closed, whence $I_0 = I = [0, 1]$.

If $v(t) \in \Sigma \times (e^{-c}, e^c)$ and $t \in [1/2, 1]$ then $r(t)$ satisfies the equation

$$\dot{r}(t) = -\dot{\vartheta}(t) \frac{\dot{\rho}_{\vartheta(t)}(x(t))}{\rho_{\vartheta(t)}(x(t))} \cdot r(t).$$

Set

$$C := \max \left\{ \left| \frac{\dot{\rho}_t(x)}{\rho_t(x)} \right| : (t, x) \in [0, 1] \times \Sigma \right\}.$$

Since $0 \leq \dot{\vartheta} \leq 4$, we see that for $t \in I_0 \cap [1/2, 1]$ it holds that

$$e^{-4C} \leq r(t) \leq e^{4C}.$$

In particular, provided $c > c_0 := 8C$ then we have that if $v(t) \in \Sigma \times (e^{-c}, e^c)$ then actually $v(t) \in \Sigma \times (e^{-c/2}, e^{c/2})$. This shows that I_0 is closed as required. \blacksquare

As an immediate corollary, we obtain:

Corollary 2.6. *For $c > c_0$ the critical point equation and the critical values for critical points of \mathcal{A}_c are independent of c . In fact, they agree with those of \mathcal{A} .*

Remark 2.7. It is important to note that if ψ is a local contactomorphism then the constant $c_0 = c_0(\psi)$ is invariant under the contact rescaling τ_R from Remark 2.4. More precisely, if ψ_R is the rescaling of ψ as in Remark 2.4, then $c_0(\psi) = c_0(\psi_R)$.

Fix a family $\mathbf{J} = (J_t)_{t \in S^1}$ of ω -compatible almost complex structures on X such that the restriction $J_t|_{\Sigma \times [1, \infty)}$ is of SFT-type (see [CFO10]). From \mathbf{J} we obtain an L^2 -inner product $\langle\langle \cdot, \cdot \rangle\rangle_{\mathbf{J}}$ on $\mathcal{L}X \times \mathbb{R}$ by

$$\langle\langle (\zeta, l), (\zeta', l') \rangle\rangle_{\mathbf{J}} := \int_0^1 \omega(J_t \zeta(t), \zeta'(t)) dt + ll'.$$

We denote by $\nabla_{\mathbf{J}} \mathcal{A}_c$ the gradient of \mathcal{A}_c with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{\mathbf{J}}$. Given $-\infty < a < b < \infty$ we denote by $\mathcal{M}_{\mathbf{J}}(\mathcal{A}_c)_a^b$ the set of smooth maps $u = (v, \eta) \in C^\infty(\mathbb{R}, \mathcal{L}X)$ that satisfy

$$\partial_s u + \nabla_{\mathbf{J}} \mathcal{A}_c(u(s)) = 0,$$

$$a < \mathcal{A}_c(u(s)) < b \quad \text{for all } s \in \mathbb{R}.$$

The following result is by now standard (see [AS09, AF10a]).

Proposition 2.8. *Given $-\infty < a < b < \infty$ and \mathbf{J} as above, if $c > c_0$ then there exists a compact set $K = K(c, \mathbf{J}, a, b) \subseteq X \times \mathbb{R}$ such that for all $u = (v, \eta) \in \mathcal{M}_{\mathbf{J}}(\mathcal{A}_c)_a^b$ one has*

$$u(\mathbb{R} \times S^1) \subseteq K.$$

Theorem 1.4 follows from Proposition 2.8 by arguments from [AF10a], as we now explain.

Proof of Theorem 1.4. (1.) Suppose ψ is a local contactomorphism. Fix $c > c_0$. After possibly replacing ψ by ψ_R for some R sufficiently large (see Remark 2.4) we may assume $\|H_c\| < \wp(\Sigma, \alpha)$ (where H_c is the Hamiltonian corresponding to ψ_R - note we are implicitly using Remark 2.7 here).

It follows from the proof of Theorem A in [AF10a] that there exists a critical point (v, η) of \mathcal{A}_c with $|\eta| \leq \|H_c\|$. Thus the translated point is necessarily a genuine translated point of ψ_R , that is, $x(0) \in \text{int}(\mathfrak{S}(\psi_R))$. Thus ψ_R , and hence ψ , has a translated point in the interior of its support.

(2.) It follows directly from Proposition 2.8 that the **Rabinowitz Floer homology** $\text{RFH}(\mathcal{A}_c)_a^b$ is well defined for $c > c_0$. Here we are using the Rabinowitz action functional is generically Morse. This is proved exactly as in [AF10a, Appendix A]. The only difference is that we are working with a more restrictive class of Hamiltonian perturbations (i.e. rather than arbitrary Hamiltonians with time support in $[1/2, 1]$, here we work only with contact Hamiltonians which have been reparametrized to have time support in $[1/2, 1]$), but the proof still goes through. In fact, the only place in the proof given in [AF10a] where the fact that we are working with a more restrictive class of Hamiltonian perturbations could onceivably cause problems is in deducing Equation (A.21) from Equation (A.18) on [AF10a, p95]. Nevertheless, the reader may check that even in our more restricted setting Equation (A.21) does indeed follow from Equation (A.18).

Moreover $\text{RFH}(\mathcal{A}_c)_a^b$ is independent of the choice of $c > c_0$. Thus it makes sense to define $\text{RFH}(\{\psi_t\}, \Sigma, X)$ via

$$\text{RFH}(\{\psi_t\}, \Sigma, X) := \varinjlim_{a \downarrow -\infty} \varprojlim_{b \uparrow \infty} \text{RFH}_*(\mathcal{A}_c)_a^b.$$

See [AF10a] for more information. In fact, by arguing as in [AF10a, Theorem 2.16], we have

$$\text{RFH}(\{\psi_t\}, \Sigma, X) \cong \text{RFH}(\Sigma, X),$$

where $\text{RFH}(\Sigma, X)$ denotes the Rabinowitz Floer homology of (Σ, X) , as defined in [CF09]. The second statement of Theorem 1.4 now follows from Lemma 2.2 and Corollary 2.6, exactly as in [AF10a, Theorem C].

(3.) The third statement in Theorem 1.4 follows from the Main Theorem in [Kan10].

(4.) The fact that generically one doesn't find translated points on closed Reeb orbits when $\dim \Sigma \geq 3$ is proved exactly as in [AF08, Theorem 3.3] (as in Statement (2) above, the fact that

we are working with a more restrictive class of Hamiltonian perturbations does not cause complications here).

(5.) Finally, the fifth statement is proved by arguing as follows. Fix $k \in \{2, 3, \dots, \infty\}$. Denote by \mathcal{H}^k the class of C^k contact Hamiltonians H , reparametrized so that $H(t, \cdot) = 0$ for $t \in [0, 1/2]$, which additionally have possibly been cutoff outside of a neighborhood of $\Sigma \times \{1\}$.

Recall given (any) two Hamiltonians K_1, K_2 , the composition $K_1 \# K_2$ is defined by

$$(K_1 \# K_2)(t, p) := K_1(t, p) + K_2(t, (\phi_t^{K_1})^{-1}(p)).$$

Denote by $K^{\#m} := K \# \dots \# K$ (m times). Note that if $H \in \mathcal{H}^k$ then $H^{\#m} \in \mathcal{H}^k$ for all $m \in \mathbb{N}$.

Given $H \in \mathcal{H}^k$, we denote by \mathcal{A}_H the Rabinowitz action functional

$$\mathcal{A}_H(v, \eta) = \int_0^1 v^* \lambda - \eta \int_0^1 \chi(t) F_0(v) dt - \int_0^1 H(t, v) dt$$

(so that the functional \mathcal{A}_c would now be written as \mathcal{A}_{H_c}).

Let $\mathcal{L} = W^{1,2}(S^1, X)$ and let \mathcal{E} denote the Banach bundle over \mathcal{L} with fibre $\mathcal{E}_v := L^2(S^1, v^* TX)$.

Fix $l, m \in \mathbb{N}$. We now define a section

$$\sigma : \mathcal{L} \times \mathbb{R} \times \mathcal{L} \times \mathbb{R} \times \mathcal{H}^k \rightarrow \mathcal{E}^\vee \times \mathbb{R} \times \mathcal{E}^\vee \times \mathbb{R}$$

by

$$\sigma(v, \eta, w, \tau, H) := (\mathrm{d}\mathcal{A}_{H^{\#l}}(v, \eta), \mathrm{d}\mathcal{A}_{H^{\#m}}(w, \tau)).$$

Let $\mathcal{M} := \sigma^{-1}(\text{zero section})$, so that

$$\mathcal{M} = \mathrm{Crit}(\mathcal{A}_{H^{\#l}}) \times \mathrm{Crit}(\mathcal{A}_{H^{\#m}}).$$

The vertical derivative of σ ,

$$D\sigma(v, \eta, w, \tau, H) := T_v \mathcal{L} \times \mathbb{R} \times T_w \mathcal{L} \times \mathbb{R} \times \mathcal{H}^k \rightarrow \mathcal{E}_v^\vee \times \mathbb{R} \times \mathcal{E}_w^\vee \times \mathbb{R}$$

is given by

$$(\hat{v}, \hat{\eta}, \hat{w}, \hat{\tau}, \hat{H}) \mapsto \left(\mathbf{H}_{\mathcal{A}_{H^{\#l}}}(v, \eta)((\hat{v}, \hat{\eta}, \hat{H}), \bullet), \int_0^1 \hat{H}^{\#l}(t, v) dt, \mathbf{H}_{\mathcal{A}_{H^{\#m}}}(w, \tau)((\hat{w}, \hat{\tau}, \hat{H}^{\#m}), \bullet), \int_0^1 \hat{H}^{\#m}(t, w) dt \right),$$

where, e.g. $\mathbf{H}_{\mathcal{A}_{H^{\#l}}}(v, \eta)$ denotes the Hessian of $\mathcal{A}_{H^{\#l}}$ at the critical point (v, η) .

In general one cannot hope that $D\sigma(v, \eta, w, \tau, H)$ is surjective. However if we set

$$\mathcal{B} := \{(v, \eta, w, \tau, H) : v(t) \neq w(t) \text{ for all } t \in [1/2, 1]\}$$

then $D\sigma(v, \eta, w, \tau, H)$ is surjective for $(v, \eta, w, \tau, H) \in \mathcal{M}^* := \mathcal{B} \cap \mathcal{M}$. In fact, if

$$\mathcal{V}(v, \eta, w, \tau, H) \subseteq T_v \mathcal{L} \times \mathbb{R} \times T_w \mathcal{L} \times \mathbb{R} \times \mathcal{H}^k$$

denotes the subspace of quintuples $(\hat{v}, \hat{\eta}, \hat{w}, \hat{\tau}, \hat{H})$ satisfying $\hat{v}(0) = \hat{w}(1/2) = 0$ then the fact that $D\sigma(v, \eta, w, \tau, H)|_{\mathcal{V}(v, \eta, w, \tau, H)}$ is surjective for all $(v, \eta, w, \tau, H) \in \mathcal{M}^*$ can be proved exactly as in [AF10a, Proposition A.2]. Now define

$$\phi_{\mathrm{eval}} : \mathcal{L} \times \mathbb{R} \times \mathcal{L} \times \mathbb{R} \times \mathcal{H}^k \rightarrow \Sigma \times \Sigma$$

by

$$\phi_{\mathrm{eval}}(v, \eta, w, \tau, H) := (v(0), w(1/2)).$$

Then from the claim it follows that $\phi_{\mathrm{eval}}|_{\mathcal{M}^*}$ is a submersion for generic $H \in \mathcal{H}^k$ (see for instance [AF08, Lemma 3.5]). Thus there exists a generic set $\mathcal{H}_{l,m}^k \subseteq \mathcal{H}^k$ for which the following property holds: if (v, η) is a critical point of $\mathcal{A}_{H^{\#l}}$, and (w, τ) is a critical point of $\mathcal{A}_{H^{\#m}}$, which satisfies

$v(t) \neq w(t)$ for all $t \in [1/2, 1]$, then $v(0) \neq w(1/2)$. Set $\mathcal{H}^* := \bigcap_{k,l,m \geq 2} \mathcal{H}_{l,m}^k$.

Claim: Suppose $H \in \mathcal{H}^*$. Set $\varphi := \phi_1^H$. Then for all pairs (l, m) of positive integers, φ^l and φ^m do not have any common leaf-wise intersection points.

To prove the claim we argue by contradiction. Without loss of generality assume $l \leq m$, and suppose $x \in \Sigma$ is a common leaf-wise intersection point of φ^l and φ^m . Thus there exists $\eta, \tau \in \mathbb{R}$ such that

$$\varphi^l(x) = \phi_\eta^\alpha(x), \quad \varphi^m(x) = \phi_\tau^\alpha(x).$$

Then

$$\varphi^{m-l}(\varphi^l(x)) = \phi_{\tau-\eta}^\alpha(\varphi^l(x)),$$

so $\varphi^l(x)$ is a leaf-wise intersection point of φ^{m-l} . Let $(v, -\eta) \in \text{Crit}(\mathcal{A}_{H\#l})$ and $(w, -\tau + \eta) \in \text{Crit}(\mathcal{A}_{H\#m})$ denote the critical points of $\mathcal{A}_{H\#l}$ and $\mathcal{A}_{H\#m}$ corresponding to x and $\varphi^l(x)$ respectively, so that $v(0) = \varphi^l(x)$ and $v(1/2) = x$, and $w(0) = \varphi^m(x)$ and $w(1/2) = \varphi^l(x) = v(0)$. By construction $v(t) \neq w(t)$ for all $t \in [1/2, 1]$, and this gives the desired contradiction. ■

References

- [AF08] P. Albers and U. Frauenfelder, *Infinitely many leaf-wise intersections on cotangent bundles*, 2008, arXiv:0812.4426, to appear in *Expositiones Mathematicae*.
- [AF10a] ———, *Leaf-wise intersections and Rabinowitz Floer homology*, J. Topol. Anal. **2** (2010), no. 1, 77–98.
- [AF10b] ———, *Rabinowitz Floer homology: A survey*, 2010, arXiv:1001.4272, to appear in *Proceedings of the SPP Global Differential Geometry*.
- [AF11] ———, *A variational approach to Givental's nonlinear Maslov index*, 2011, arXiv:1102.3627.
- [AS09] A. Abbondandolo and M. Schwarz, *Estimates and computations in Rabinowitz-Floer homology*, J. Topol. Anal. **1** (2009), no. 4, 307–405.
- [CF09] K. Cieliebak and U. Frauenfelder, *A Floer homology for exact contact embeddings*, Pacific J. Math. **293** (2009), no. 2, 251–316.
- [CFO10] K. Cieliebak, U. Frauenfelder, and A. Oancea, *Rabinowitz Floer homology and symplectic homology*, Ann. Sci. Éc. Norm. Supér. (4) **43** (2010), no. 6, 957–1015.
- [Che96] Yu. V. Chekanov, *Critical points of quasifunctions, and generating families of Legendrian manifolds*, Funktsional. Anal. i Prilozhen. **30** (1996), no. 2, 56–69, 96.
- [Kan10] J. Kang, *Survival of infinitely many critical points for the Rabinowitz action functional*, J. Mod. Dyn. **4** (2010), no. 4, 733–739.
- [San11a] S. Sandon, *A Morse estimate for translated points of contactomorphisms of spheres and projective spaces*, 2011, arXiv:1110.0691.
- [San11b] ———, *On iterated translated points for contactomorphisms of \mathbb{R}^{2n+1} and $\mathbb{R}^{2n} \times S^1$* , 2011, arXiv:1102.4202, to appear in *International Journal of Mathematics*.

PETER ALBERS, MATHEMATISCHES INSTITUT, WWU MÜNSTER, GERMANY

Email: `peter.albers@uni-muenster.de`

WILL J. MERRY, DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS,
UNIVERSITY OF CAMBRIDGE, ENGLAND

Email: `w.merry@dpms.cam.ac.uk`